Sec 3.6 Determinants

Last time: $\underline{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$,

Determinant $\det(\underline{A}) = |\underline{A}| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \overset{\text{DEF}}{=} ad - bc$

$\boxed{\bigstar \quad \underline{A}^{-1} \text{ defined} \iff |\underline{A}| \neq 0 \quad (\text{even for } 3\times3, 4\times4 \cdots)}$

Need something similar for 3×3, 4×4, $\cdots$

Idea: "Cross Product"

General method: "Cofactor Expansion"

— Pick any row or column to "expand along"

Ex $\underline{A} = \begin{bmatrix} 2 & 0 & 4 \\ 3 & 4 & 2 \\ 0 & 4 & -2 \end{bmatrix}$

Let's use column 1

$$|\underline{A}| = (2)(-1)^2 \underbrace{\begin{vmatrix} 4 & 2 \\ 4 & -2 \end{vmatrix}} + (3)(-1)^3 \begin{vmatrix} 0 & 4 \\ 4 & -2 \end{vmatrix}$$

a_{11}

$(-1)^{1+1}$

matrix $\underline{A}$ with row 1, col 1 deleted

$$+ (0)(-1)^4 \underbrace{\begin{vmatrix} 0 & 4 \\ 4 & 2 \end{vmatrix}}$$

a_{31}

$(-1)^{3+1}$

$\underline{A}$ with row 3, col 1 deleted.

$$= 2(4(-2) - 4(2)) - 3(0(-2) - 4(4)) + 0(0(2) - 4(4))$$

$$= -32 + 48 + 0 = \boxed{16}$$

$$A = \begin{bmatrix} 2 & 0 & 4 \\ 3 & 4 & 2 \\ 0 & 4 & -2 \end{bmatrix}$$

Now try along row 1:

$$|A| = (2)(-1)^2 \begin{vmatrix} 4 & 2 \\ 4 & -2 \end{vmatrix} + (0)(-1)^3 \begin{vmatrix} 3 & 2 \\ 0 & -2 \end{vmatrix} + \underset{a_{13}}{(4)}(-1)^4 \begin{vmatrix} 3 & 4 \\ 0 & 4 \end{vmatrix}$$

$$= 2(-8-8) + 4(12-0)$$

$$= -32 + 48$$

$$= \boxed{16}$$

"the (1,3) – minor" M_{13}

"the (1,3) – cofactor" $C_{13} = (-1)^{1+3} M_{13}$

So "cofactor expansion along row 1"

$$\longrightarrow |A| = a_{11} C_{11} + a_{12} C_{12} + a_{13} C_{13}$$

$$= a_{11} (-1)^{1+1} M_{11} + a_{12} (-1)^{1+2} M_{12} + a_{13} (-1)^{1+3} M_{13}$$

along Column 1:

$$|A| = a_{11} C_{11} + a_{21} C_{21} + a_{31} C_{31}$$

$$= a_{11} (-1)^{1+1} M_{11} + a_{21} (-1)^{2+1} M_{21} + a_{31} (-1)^{3+1} M_{31}$$

Any choice of row/column for cofactor expansion gives same value $|A|$.

(so pick one with more zeroes)

- factor $(-1)^{i+j}$ for M_{ij} gives signs $\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$

$\downarrow$

Ex $\underline{A} = \begin{bmatrix} 8 & 0 & 0 & 5 \\ 5 & 8 & 3 & -7 \\ 2 & 0 & 0 & 0 \\ 7 & 2 & 1 & 7 \end{bmatrix}$. signs $\rightarrow \begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}$

use row 3 for $|\underline{A}|$:

$$|\underline{A}| = a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33} + a_{34}C_{34}$$

$$= +2M_{31} \underbrace{- 0M_{32} + 0M_{33} - 0M_{34}}_{\text{no need to compute!}}$$

$$= 2\begin{vmatrix} 0 & 0 & 5 \\ 8 & 3 & -7 \\ 2 & 1 & 7 \end{vmatrix}$$

$$= 2\left((+)5 \begin{vmatrix} 8 & 3 \\ 2 & 1 \end{vmatrix} \right) = 10(8-6) = \boxed{20}$$

Cramer's Rule

(alt to (RREF $\underline{A}$) or $\underline{A}^{-1}$)

Solving system $\underset{n\times n}{\underline{A}}\,\underset{n\times 1}{\underline{x}} = \underset{n\times 1}{\underline{b}}$

$$\underbrace{\begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}}_{\underbrace{[\underline{c}_1 \;\; \underline{c}_2]}_{\underline{A}}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\underline{x}} = \underbrace{\begin{bmatrix} 9 \\ 8 \end{bmatrix}}_{\underline{b}}$$

- If $|\underline{A}| \neq 0$, then can compute solution via:

$$\underline{B}_1 \overset{\text{DEF}}{=} [\underline{b} \;\; \underline{c}_2] = \begin{bmatrix} 9 & 3 \\ 8 & 1 \end{bmatrix}$$

$$\underline{B}_2 = [\underline{c}_1 \;\; \underline{b}] = \begin{bmatrix} 1 & 9 \\ 2 & 8 \end{bmatrix}$$

Cramer's Rule

$$x_1 = \frac{|\underline{B_1}|}{|\underline{A}|} = \frac{\begin{vmatrix} 9 & 3 \\ 8 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix}} = \frac{-15}{-5} = 3$$

$$x_2 = \frac{|\underline{B_2}|}{|\underline{A}|} = \frac{\begin{vmatrix} 1 & 9 \\ 2 & 8 \end{vmatrix}}{\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix}} = \frac{-10}{-5} = 2.$$

Works for 3x3, ... too.

<u>Ex:</u>

$$\begin{cases} x_1 + x_2 = 4 \\ -4x_1 + 3x_3 = 0 \\ x_2 - 3x_3 = 3 \end{cases}$$

$$\underbrace{\begin{bmatrix} 1 & 1 & 0 \\ -4 & 0 & 3 \\ 0 & 1 & -3 \end{bmatrix}}_{\underline{A}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\underline{x}} = \underbrace{\begin{bmatrix} 4 \\ 0 \\ 3 \end{bmatrix}}_{\underline{b}}$$

$$x_1 = \frac{|\underline{B_1}|}{|\underline{A}|} = \frac{\begin{vmatrix} 4 & 1 & 0 \\ 0 & 0 & 3 \\ 3 & 1 & -3 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 0 \\ -4 & 0 & 3 \\ 0 & 1 & -3 \end{vmatrix}} = \cdots = \frac{1}{5}$$

Transpose : Swap rows $\longleftrightarrow$ columns

$$\underline{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad \underline{A}^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

$$\underline{B} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}_{(2\times3)} \qquad \underline{B}^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}_{(3\times2)}$$

Algebra :

- $(\underline{A}+\underline{B})^T = \underline{A}^T + \underline{B}^T$
- $(\underline{A}^T)^T = \underline{A}$
- ★ $(\underline{AB})^T = \underline{B^T A^T}$ NOT $\underline{A^T B^T}$